Journal of Applied Mechanics and Technical Physics, Vol. 37, No. 2, 1996

GROUP ANALYSIS OF THE PARTIALLY SYMMETRIZED FORM OF THE SYSTEM OF EQUATIONS OF FREE THERMOCONCENTRATIONAL CONVECTION

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UDC 532.516.539.3

Free convective structures have attracted considerable interest in recent years because this characteristic type of flows, which arises under the action of comparatively small buoyancy forces caused by minor differences in density of the fluid existing in a mass force field, is frequently encountered under natural conditions and in technological processes. It is thermoconcentrational convection that is responsible for the formation of fine structures (spatially regular in depth) of the ocean, of the atmosphere of planets and stars, of the earth's mantle, and of other geophysical systems [1]. Also, convective flows are often spatially ordered in a horizontal direction. The scales of these structures vary in extremely wide ranges, from several millimeters in the laboratory up to tens of thousand kilometers in the sun's atmosphere. The flow pattern depends on the geometry and dimensionality of the source and on the medium's stratification [2, 3]. In a homogeneous medium, the jet over a point heat source can initially be laminar but loses stability and becomes turbulent at some height. In a thermally stratified medium, the heated particles pass the neutral buoyancy level, and are then decelerated and fall on it, forming a typical mushroom-shaped jet [4]. In a medium with steady stratification of admixture, two main types of structures are observed: cells elongated along the horizontal and separated by high-gradient interlayers (a "thermohaline stair") [1, 4] and cells elongated along the vertical ("salt fingers") [4]. The types of convective structures are described in [1, 4] and the theory of convectional heat and mass transfer is given in [2].

Experimental investigations of convection over a localized (point) source [5, 6] and over an extended (linear) source [7] have shown that "salt fingers" are also present inside the cells. This extremely hinders the development of adequate theoretical models, which should necessarily take into account the nonlinearity of the equation of state for the medium, the Soret and Dufour effects, admixture and heat transfer, and dependences of the kinetic coefficients of the medium, of the thermal expansion, and of the salt compression coefficients on the state parameters.

In studies of the main convection laws, the equations of motion are usually written in the Boussinesq approximation and are additionally simplified according to the chosen model of the process. The involvement coefficients are considered constant [8, 9] and linearized systems are used [10, 11].

In analyzing a system of nonlinear nonstationary equations with complex boundary conditions, one should apply reasonably powerful mathematical methods. One of them is theoretical group analysis, which allows one to study the invariant properties of the equations of thermoconcentrational convection. Apparently, one of the first attempts at such analysis was undertaken by McKenzie [12], who has studied only crystallographic symmetry groups that are admitted by the system of equations of thermoconcentrational convection. A complete group analysis of the equations has been performed in [13]; as a result, dependences of the scales of dynamic structures that arise on the power generated by a heat source were obtained. At the same time, many questions arising in studies of thermoconcentrational convection have not been answered in [12, 13].

Institute of Problems of Mechanics, Russian Academy of Sciences, Moscow 117526. Translated from Prikladnaya Mekhanika Tekhnicheskaya Fizika, Vol. 37, No 2, pp. 14–26, March–April, 1996. Original article submitted February 24, 1995.

0021-8944/96/3702-0159 $15.00 © 1996 Plenum Publishing Corporation

The goal of this paper is to derive a partially symmetrized from for the initial nonlinear nonstationary system of the convection equations. This form is more convenient in using group analysis to determine invariant properties and in finding relationships of spatial–temporal scales of flows in a form that allows direct comparison with experiment.

Statement of the Problem. The system of equations of thermoconcentrational convection for a multicomponent medium in the Boussinesq approximation has the form [14]

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} + \mathbf{g}\Big(\sum_i \beta_i S_i - \alpha T\Big),$$

$$\frac{\partial \bar{S}_i}{\partial t} + \nabla\cdot(\bar{S}_i\mathbf{u}) = k_i\Delta\bar{S}_i + H_i(\mathbf{R},t), \qquad \frac{\partial \bar{T}}{\partial t} + \mathbf{u}\nabla\bar{T} = \chi\Delta\bar{T} + F(\mathbf{R},t),$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = \rho_0\sum_i \beta_i H_i(\mathbf{R},t), \quad \rho = \rho_0\Big(1 + \sum_i \beta_i\bar{S}_i - \alpha\bar{T}\Big), \quad \bar{S}_i = S_{0i}(z) + S_i, \tag{1}$$

$$\bar{T} = T_0(z) + T, \qquad S_{0i}(z) = S_{0i}\Big(1 - \frac{z}{\Lambda_i}\Big), \qquad T_0(z) = T_0\Big(1 + \frac{z}{\Lambda_T}\Big).$$

Here p is the pressure minus hydrostatic pressure, which is normalized by ρ_0; ρ and ρ_0 are the total density of the medium and the density of the medium in the absence of admixtures; $\mathbf{u}$ is the velocity field; $\bar{T}$, $T_0(z)$, and T are the total, stratifying, and excess temperatures; $\bar{S}_i$, $S_{0i}(z)$, and S_i are the total, stratifying, and additional concentration of the ith admixture; ν, χ, and k_i are the kinematic viscosity, thermal conductivity, and diffusivity of the ith admixture; α and β_i are the coefficients of thermal expansion and of salt compression for the ith admixture; $F(\mathbf{R},t)$ and $H_i(\mathbf{R},t)$ are the sources of heat and of the ith admixture.

Since the aim of this work is a group analysis of system (1), we may not now specify the initial and boundary conditions.

The solenoidal part of the velocity field makes the main contribution to the temperature and admixture transfer due to the convective terms $\mathbf{u}\nabla T$ and $\mathbf{u}\nabla S_i$. Therefore it is convenient to divide $\mathbf{u}$ into potential and solenoidal components. We cannot simply assume that the potential component of the velocity equals zero, because it describes expansion of an element of the medium and, thus, gives rise to buoyancy forces, which participate in the formation of convective flow. A linear combination of some equations of system (1) (without using the Navier–Stokes equation), in view of the fact that in real situations $\alpha\bar{T} \ll 1$, yields

$$\nabla\cdot\mathbf{u} \approx \Delta\Big(\chi\alpha T - \sum_i \beta_i S_i\Big) + \alpha F. \tag{2}$$

Using (2), it is possible to pass over from the analysis (solution) of system (1) to an analysis (solution) of an approximate system, in which the continuity equation is replaced by the equation $\nabla\cdot\mathbf{u} = 0$. Then the solution of (1) will be reduced to the consecutive solution of the approximate system followed by substitution into (2) of necessary temperature and admixture distributions and by determination of the potential part of the velocity field. The sum of solutions of the approximate system and Eq. (2) gives an approximate solution of system (1).

The following step in preparing system (1) to analysis involves the preliminary symmetrization of the equations of thermoconcentrational convection with respect to the variables T and S_i.

Introduction of Generalized Disturbances of Density. In [2, 13, 14] combinations of the form $\beta S - \alpha T$, $\chi\beta S - k_S\alpha T$, $\chi\alpha T - k_S\beta S$, etc., were chosen. It can be noticed that the first combination enters in the Navier–Stokes equation of system (1), and the third, in relationship (2). The transition from the system of variables (T, S) to a system of any two of the above-mentioned combinations brings about not only a change in the notation of the fundamental equations but also the appearance of combinations of the type $\chi^2\alpha T - k_S^2\beta S$, $k_S^2\alpha T - \chi^2\beta S$, etc. As a result, it becomes possible to pass over to a new, more complex pair (or a system in the case of multicomponent media) of combinations of physical fields. In doing so, we have no assurance that the combinations chosen as the main are the most optimal combinations. For all the abundance of variations, nevertheless, there is a possibility of describing all actually combinations that arise by means of a unified approach. This is achieved by introducing generalized disturbances of density.

First we consider a combination of the form

$$m_\lambda = \alpha T \left(\frac{\chi^N}{\prod\limits_{i=1}^{N} k_i} \right)^{\lambda/(N+1)} - \sum_{j=1}^{N} \beta_j S_j \left(\frac{k_j^N}{\chi \prod\limits_{i \neq j}^{N} k_i} \right)^{\lambda/(N+1)} \tag{3}$$

(λ is an arbitrary number). In this case the following relation holds:

$$\varphi_{\lambda_1\lambda_2\ldots\lambda_N\lambda_{N+1}} m_{\lambda_{N+2}} \pm \varphi_{\lambda_2\lambda_3\ldots\lambda_{N+1}\lambda_{N+2}} m_{\lambda_1} + \varphi_{\lambda_3\lambda_4\ldots\lambda_{N+2}\lambda_1} m_{\lambda_2} \pm \ldots + \varphi_{\lambda_{N+2}\lambda_1\ldots\lambda_N} m_{\lambda_{N+1}} = 0, \tag{4}$$

where

$$\varphi_{\lambda_i\lambda_j\ldots\lambda_k\lambda_n} = \det \begin{vmatrix} \mu^{\lambda_i} & \mu_1^{\lambda_i} & \ldots & \mu_N^{\lambda_i} \\ \mu^{\lambda_j} & \mu_1^{\lambda_j} & \ldots & \mu_N^{\lambda_j} \\ \vdots & \vdots & & \vdots \\ \mu^{\lambda_n} & \mu_1^{\lambda_n} & \ldots & \mu_N^{\lambda_n} \end{vmatrix}; \quad \mu = \left(\frac{\chi^N}{\prod\limits_{i=1}^{N} k_i} \right)^{1/(N+1)}; \quad \mu_j = \left(\frac{k_j^N}{\chi \prod\limits_{i \neq j}^{N} k_i} \right)^{1/(N+1)}$$

A distinctive feature of combinations (3) is that they describe all the possible combinations of T and S_i obtained in system (1) by linear combination of its equations. If the N value is odd, only the plus signs in (4) are taken, and if this value is even, the signs are alternated. The matrix composed of the elements $\varphi_{\lambda_i\lambda_j\ldots\lambda_n}$, is antisymmetric, i.e., with an even rearrangement of the indices, the sign of the element of the matrix does not change, and with an odd rearrangement the sign changes. If even two indices coincide, the element vanishes.

From (4) it is evident that choosing arbitrarily $N+1$ values of the index λ, we obtain a system for $N+1$ functions $\{m_\lambda\}$, which describe any generalized disturbance of density. Without specifying the chosen values of λ_i, we can express, using (3), the temperature and concentration of admixtures in terms of the system $\{m_\lambda\}$. Substituting the result into system (1), we obtain the equations of thermoconcentrational convection in a form that involves any combination of T and S_i in the most general form without specification, which can be carried out after the system is solved in the general form.

Partially Symmetrized System of Equations of Thermoconcentrational Convection in a Medium with One Admixture. It is not difficult to transform the variables T and $\{S_i\}$ in system (1) to the variables $\{m_\lambda\}$ using relationship (3). For large N, however, this involves cumbersome manipulations; therefore, for simplicity, we consider a medium with only one admixture — salinity. Then the system of generalized density disturbances has the form

$$m_\lambda = (\chi/k_S)^{\lambda/2} \alpha T - (k_S/\chi)^{\lambda/2} \beta S$$

(k_S is the diffusion coefficient of salt).

Choosing two functions m_γ and m_δ with arbitrary indices γ and δ, expressing T and S in terms of these functions, and substituting the result into the approximate variant of system (1), we obtain a system of equations of the form

$$\begin{gathered} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \mathbf{g}\left(a_\gamma m_\gamma + a_\delta m_\delta\right), \\ \frac{\partial m_\gamma}{\partial t} + \mathbf{u} \cdot \nabla m_\gamma + w \left(\frac{T^\gamma}{\Lambda_T} + \frac{S^\gamma}{\Lambda_S} \right) = k_\gamma \Delta m_\gamma + k_{\delta\gamma} \Delta m_\delta + Q_\gamma, \\ \frac{\partial m_\delta}{\partial t} + \mathbf{u} \cdot \nabla m_\delta + w \left(\frac{T^\delta}{\Lambda_T} + \frac{S^\delta}{\Lambda_S} \right) = k_\delta \Delta m_\delta + k_{\gamma\delta} \Delta m_\gamma + Q_\delta, \quad \nabla \cdot \mathbf{u} = 0. \end{gathered} \tag{5}$$

Here $\mathbf{u} = (u, v, w)$; Λ_T and Λ_S are the temperature and salt stratification scales; $\gamma \neq \delta$;

$$T^\lambda = (\chi/k_S)^{\lambda/2}\,\alpha T_0;\quad S^\lambda = (k_S/\chi)^{\lambda/2}\,\beta S_0;\quad Q_\lambda = (\chi/k_S)^{\lambda/2}\,\alpha F - (k_S/\chi)^{\lambda/2}\,\beta H;\quad \lambda = \gamma,\delta;$$

$$a_\gamma = \varphi_{0\delta}/\varphi_{\gamma\delta};\quad a_\delta = \varphi_{0\gamma}/\varphi_{\delta\gamma};\quad k_{\gamma\delta} = \frac{\chi - k_S}{\varphi_{\gamma\delta}};\quad k_\gamma = \left(\chi\,(\chi/k_S)^{(\gamma-\delta)/2} - k_S\,(k_S/\chi)^{(\gamma-\delta)/2}\right)/\varphi_{\gamma\delta};$$

$$k_\delta = \left(k_S\,(\chi/k_S)^{(\gamma-\delta)/2} - \chi\,(k_S/\chi)^{(\gamma-\delta)/2}\right)/\varphi_{\gamma\delta};\quad \varphi_{\alpha\beta} = (\chi/k_S)^{(\alpha-\beta)/2} - (k_S/\chi)^{(\alpha-\beta)/2}.$$

System (5) involves only the ratios and differences of the kinetic coefficients χ and k_S, i.e., the system is symmetrized with respect to them. The coefficients a_γ and a_δ and k_γ, k_δ, $k_{\gamma\delta}$, and $k_{\delta\gamma}$ are mutually symmetric, and (5) is symmetric with respect to them. The generalized sources Q_λ also have a symmetric form. In addition, (5) is symmetric with respect to the generalized density disturbances. In contrast to (1) in which there is a distinction between T and S, the functions m_γ and m_δ in (5) are absolutely equivalent, i.e., system (5) is a system of symmetrized form. This symmetrization, however, is partial because it does not involve symmetrization between ν and (χ, k_S) and also between the vector fields $\mathbf{u}$ and $(\nabla m_\gamma, \nabla m_\delta)$.

Lie Group of Special Form. The standard group analysis of system (5) consists in finding a continuous group of the form [15]

$$G\cdot\partial = A\partial_x + B\partial_y + C\partial_z + D\partial_t + U\partial_u + V\partial_v + W\partial_w + E\partial_p + M_\gamma\partial_{m_\gamma} + M_\delta\partial_{m_\delta}, \tag{6}$$

which preserves the invariance of (5) with respect to transformations of the differential $\{x, y, z, t\}$ and field $\{u, v, w, p, m_\gamma, m_\delta\}$ variables that are described by relationship (6). Use of this transformation, however, does not permit one to find the invariant properties of system (5) in relation to changes in the stratification scales Λ_T and Λ_S, which are of great interest. In this work, therefore, a group of special form is used:

$$\widetilde{G}\cdot\partial = G\cdot\partial + L_T\partial_{\Lambda_T} + L_S\partial_{\Lambda_S}. \tag{7}$$

In this case, the rule whereby the first and second continuations of the group $\widetilde{G}\cdot\partial$ are sought for is that A, B, C, D, U, V, W, E, M_γ, M_δ, L_T, and L_S are considered functions of all variables of the problem, including Λ_T and Λ_S, and differential but field variables as group analysis variables are not functions of Λ_T and Λ_S, i.e., Λ_T and Λ_S can be considered parametric variables. Of course, the fields of u, v, w, p, m_γ, and m_δ as solutions of system (5) are functions of x, y, z, t and Λ_T, Λ_S, but in terms of group analysis field variables have a somewhat different meaning.

In the analysis of (5) we first seek a group of differential operators of this system, i.e., a group of the system in which the functions of the sources Q_γ and Q_δ are set to zero.

Application of special group (7) to this reduced system allows determination of the infinite-dimensional group of transformations $\widetilde{G}\cdot\partial$, whose generators have the form

$$\begin{gathered}
G_1\cdot\partial = x\partial_x + y\partial_y + z\partial_z + 2t\partial_t - u\partial_u - v\partial_v - w\partial_w - 2p\partial_p - 3m_\gamma\partial_{m_\gamma} - 3m_\delta\partial_{m_\delta} + 4\Lambda_T\partial_{\Lambda_T} + 4\Lambda_S\partial_{\Lambda_S},\\
G_2\cdot\partial = y\partial_x - x\partial_y + v\partial_u - u\partial_v,\quad G_3\cdot\partial = A\partial_x + \frac{\partial A}{\partial t}\partial_u - \frac{\partial^2 A}{\partial t^2}x\partial_p,\quad G_4\cdot\partial = B\partial_y + \frac{\partial B}{\partial t}\partial_v - \frac{\partial^2 B}{\partial t^2}y\partial_p,\\
G_5\cdot\partial = C\partial_z + \frac{\partial C}{\partial t}\partial_w - \frac{\partial^2 C}{\partial t^2}z\partial_p - C\left(\frac{T^0}{\Lambda_T} + \frac{S^0}{\Lambda_S}\right)\left(gz\partial_p + k^{\gamma/2}\partial_{m_\gamma} + k^{\delta/2}\partial_{m_\delta}\right),\quad G_6\cdot\partial = D\partial_t,\\
G_7\cdot\partial = E\partial_p,\quad G_8\cdot\partial = gz\partial_p + k^{\gamma/2}\partial_{m_\gamma} + k^{\delta/2}\partial_{m_\delta},\quad G_9\cdot\partial = gz\partial_p + k^{-\gamma/2}\partial_{m_\gamma} + k^{-\delta/2}\partial_{m_\delta},\\
G_{10}\cdot\partial = \frac{gz^2}{2}\partial_p + k^{\gamma/2}z\partial_{m_\gamma} + k^{\delta/2}z\partial_{m_\delta} + \frac{\Lambda_T^2}{T^0}\partial_{\Lambda_T},\quad G_{11}\cdot\partial = \frac{gz^2}{2}\partial_p + k^{-\gamma/2}z\partial_{m_\gamma} + k^{-\delta/2}z\partial_{m_\delta} + \frac{\Lambda_S^2}{S^0}\partial_{\Lambda_S},\\
G_\lambda\cdot\partial = gJ(\lambda, z, t)\partial_p + k^{\pm\gamma/2}J'_z\partial_{m_\gamma} + k^{\pm\delta/2}J'_z\partial_{m_\delta} + \frac{\Lambda_\lambda^2}{R_\lambda^0}J''_{zz}\partial_{\Lambda_\lambda}.
\end{gathered}\tag{8}$$

Here A, B, C, and E are arbitrary functions of t, Λ_T, and Λ_S and D is an arbitrary function of Λ_T and Λ_S;

$$\lambda = \begin{Bmatrix} \chi \\ k_S \end{Bmatrix}; \qquad \Lambda_\lambda = \begin{Bmatrix} \Lambda_T \\ \Lambda_S \end{Bmatrix}; \qquad R_\lambda^0 = \begin{Bmatrix} T^0 \\ S^0 \end{Bmatrix}; \qquad k = \chi/k_S;$$

$$J(\lambda, z, t) = \int_0^z \Phi_\lambda(\zeta, t)\, d\zeta, \quad \text{where} \quad \Phi_\lambda(\zeta, t) = \int_{-\infty}^{+\infty} \varphi(\xi) e^{i\xi\zeta - \xi^2 \lambda t}\, d\xi$$

is a solution of the equation $\Phi'_{\lambda_t} - \lambda \Phi''_{\lambda_{\zeta\zeta}} = 0$, and $\varphi(\xi)$ is an arbitrary function such that

$$|\Phi_\lambda| < \infty, \qquad |\Phi'_{\lambda_\zeta}| < \infty, \qquad |\Phi''_{\lambda_{\zeta\zeta}}| < \infty.$$

Thus for the upper value of λ in braces we take the upper values of Λ_λ and R_λ^0 and the plus signs in the expressions for $G_\lambda \cdot \partial$; conversely, for the lower value of λ, we take the lower values of Λ_λ and R_λ^0 and minus signs.

It should be noted that multiplication of any generator $G_i \cdot \partial$ by an arbitrary function of the stratification scales Λ_T and Λ_S yields a new generator, which, as applied to system (5), leaves it invariant.

Some generators of system (8) have a reasonably clear physical meaning. Thus, the generator $G_1 \cdot \partial$ describes the scale-invariant properties of solutions of system (5); $G_2 \cdot \partial$ points to the cylindrical symmetry of convective flow about the z axis; $G_3 \cdot \partial, G_4 \cdot \partial$ and $G_5 \cdot \partial$ are generators of Galilean transforms, which take into account changes in the pressure field of the medium; $G_6 \cdot \partial$ points to the invariance of solutions with respect to time shifts; $G_7 \cdot \partial$ reflects the fact that the forces generated in the medium by the pressure are purely potential; and $G_8 \cdot \partial$ and $G_9 \cdot \partial$ indicate that if the generalized densities receive constant additives, the isobars in the medium are displaced along the z axis.

To elucidate the physical meaning of the generators $G_{10} \cdot \partial$ and $G_{11} \cdot \partial$, we integrate their linear combination

$$G \cdot \partial = aG_{10} \cdot \partial + bG_{11} \cdot \partial, \tag{9}$$

where a and b are arbitrary numbers.

Integration of (9) and transformation to the variables T and S leads to explicit form of transformations that leave system (5) invariant:

$$T^* = T + a\varepsilon z/\alpha, \quad S^* = S - b\varepsilon z/\beta, \quad p^* = p + \varepsilon(a+b) g z^2/2, \tag{10}$$

$$\Lambda_T^* = \frac{\Lambda_T}{1 - a\varepsilon \dfrac{\Lambda_T}{\alpha T_0}}, \qquad \Lambda_S^* = \frac{\Lambda_S}{1 - b\varepsilon \dfrac{\Lambda_S}{\beta S_0}}$$

(the asterisk denotes the new variables and ε is an arbitrary parameter of the transformations).

From (10) it is seen that with appropriate changes in the temperature and salt stratification scales (relationships for Λ_T^* and Λ_S^*) and with additional stratifying additives to the temperature and salinity fields (relationship for T^* and S^*) such that the scales of these stratifications are inversely proportional to the coefficients of thermal expansion and of salt contribution to density, the pressure field isobars (relationship for p^*) either crowd together (at $a + b > 0$) or are thinned out (for $a + b < 0$). For $a + b = 0$, the pressure field does not change, nor does the field of density disturbances change, since in this case

$$\rho^* \equiv \beta S^* - \alpha T^* = \beta S - \alpha T - (a+b) z\varepsilon = \beta S - \alpha T \equiv \rho.$$

It should be noted that the total temperature $\bar{T}$ and total salinity $\bar{S}$ distributions are invariants of this transformation, since

$$\bar{T}^* \equiv T_0^*(z) + T^* = T_0\left(1 + \frac{z}{\Lambda_T^*}\right) + T + \frac{a\varepsilon}{\alpha} z = T_0\left(1 + \frac{z}{\Lambda_T}\left(1 - \frac{a\varepsilon}{\alpha T_0}\Lambda_T\right)\right) + T + \frac{a\varepsilon}{\alpha} z = T_0\left(1 + \frac{z}{\Lambda_T}\right) + T \equiv \bar{T}.$$

A similar sequence of relationships holds for $\bar{S}$. This means that the concentration of isobars has a purely hydrostatic character [this is confirmed by the type of the additive $(a+b) g z^2 \varepsilon/2$ to the pressure field] and does not influence the character of convective flow, which is in complete agreement with rejection of the hydrostatic term of pressure in system (1).

It seems impossible to elucidate the physical meaning of the infinite set of generators $G_\lambda \cdot \partial$ in the most general form, since the form of the function $\varphi(\xi)$ is arbitrary. To substantiate this statement, we consider a simple special case. For this, we use a linear combination of two generators:

$$G \cdot \partial = aG_\chi \cdot \partial + bG_{k_S} \cdot \partial \tag{11}$$

such that $\varphi(\xi) \equiv 1$ for both $G_\chi \cdot \partial$ and for $G_{k_S} \cdot \partial$. In this case, we have

$$\Phi_\chi(\zeta, t) = \sqrt{\frac{\pi}{\chi t}}\mathrm{e}^{-\frac{\zeta^2}{4\chi t}}, \qquad \Phi_{k_S}(\zeta, t) = \sqrt{\frac{\pi}{k_S t}}\mathrm{e}^{-\frac{\zeta^2}{4k_S t}}.$$

The explicit form of transformations that correspond to (11) is

$$\begin{gathered} p^* = p + \pi g\varepsilon\left[a\,\mathrm{erf}\left(\frac{z}{2\sqrt{\chi t}}\right) + b\,\mathrm{erf}\left(\frac{z}{2\sqrt{k_S t}}\right)\right], \quad T^* = T + \frac{a\varepsilon}{\alpha}\sqrt{\frac{\pi}{\chi t}}\mathrm{e}^{-\frac{z^2}{4\chi t}}, \\ S^* = S - \frac{b\varepsilon}{\beta}\sqrt{\frac{\pi}{k_S t}}\mathrm{e}^{-\frac{z^2}{4k_S t}}, \quad \frac{1}{\Lambda_T^*} = \frac{1}{\Lambda_T} + \frac{a\varepsilon}{T^0}\frac{z\sqrt{\pi}}{2(\chi t)^{3/2}}\mathrm{e}^{-\frac{z^2}{4\chi t}}, \quad \frac{1}{\Lambda_S^*} = \frac{1}{\Lambda_S} + \frac{a\varepsilon}{S^0}\frac{z\sqrt{\pi}}{2(k_S t)^{3/2}}\mathrm{e}^{-\frac{z^2}{2k_S t}}. \end{gathered} \tag{12}$$

As can be seen from (12), the transformations from the old coordinates to the new ones are of an additive diffusive character in the presence of two split dynamic layers (temperature and salt), whose dimensions vary according to the laws $\sqrt{\chi t}$ and $\sqrt{k_S t}$, respectively. It is evident that the corrections to the temperature field and to the gradient of the initial temperature-stratification distribution depend on the thermal diffusivity, while corrections to the salinity and to the gradient of the initial salt-stratification distribution depend on the diffusion coefficient of the salt.

The obtained result is directly related to the stability of experimental data on thermoconcentrational convection. The fact is that if an experimental basin is filled layerwise, after the accomplishment of the procedure, some time should be allowed for smoothing of the sharp density gradients on the boundaries of the layers and for formation of stratification with a constant gradient. Ideally, to achieve a constant gradient, an infinitely long period of time should be allowed to elapse. Since this is impossible in real situations, one has to perform the experiments in the presence of diffusive processes, using diffusive corrections to the constant density gradient of the initial expected stratification. At the same time, from (12) it follows that Eqs. (5) of thermoconcentrational convection are invariant with respect to variation in fields due to similar corrections. This means that the fundamental properties of flows are stable against diffusive disturbances of density distributions in experimental basins and under natural conditions.

Here it should be emphasized once again that the properties of the generators $G_\chi \cdot \partial$ and $G_{k_S} \cdot \partial$ are not restricted to transformations (12) but are determined by the form of the function $\varphi(\xi)$ in each particular case.

Space-Invariant Properties of Convective Flows. Group analysis results (8) are obtained from the initial convection equations with the temperature and salt source functions equal to zero. Hence, the generators $G_i \cdot \partial$ give rise to the so-called groups G_D of differential operators. Imposition of the invariance condition of the source functions with respect to the operators $G_i \cdot \partial$ and allowance for the boundary conditions reduce the number of allowable operators. We further consider only those situations in which the salt-source function is identically equal to zero, which allows direct comparison with the experimental results of [3, 5, 6] to be made.

Irrespective of the type of fluid stratification (salt, temperature, or both simultaneously) the point heat source $F(\mathbf{R}, t) = F_0\delta(x)\delta(y)\delta(z)\theta(t)$ admits the generator $G_2 \cdot \partial$, which means that the arising flow has cylindrical symmetry.

The flow structures for a horizontal $[F(\mathbf{R}, t) = F_0\delta(y)\delta(z)\theta(t)$ or $F_0\delta(x)\delta(z)\theta(t)]$ and vertical $[F(\mathbf{R}, t) = F_0\delta(x)\delta(y)\theta(t)]$ linear sources are invariant with respect to transfers along the sources (generators $G_3 \cdot \partial \ldots G_5 \cdot \partial$ for $A_t' = B_t' = C_t' \equiv 0$), respectively.

The flow from a vertical plane source $F(\mathbf{R},t) = F_0\delta(y)\theta(t)$ displays invariant properties with respect to transfers that are parallel to the source plane (generators $G_3 \cdot \partial$ and $G_5 \cdot \partial$ for $A'_t = C'_t \equiv 0$).

A horizontal plane source $[F(\mathbf{R},t) = F_0\delta(z)\theta(t)]$ admits the generators $G_2 \cdot \partial \ldots G_4 \cdot \partial$ for $A'_t = B'_t \equiv 0$, which means the occurrence of structures that are invariant with respect to transfers and turns in a plane parallel to the source plane, which includes the symmetry of Benard's cells. All the above-mentioned invariant properties of flows have been observed experimentally [3, 5, 6].

None of the types of sources admits the generator $G_6 \cdot \partial$, which implies the invariance of flow with respect to shifts in time, which, in turn, emphasizes the transient character of formation of convective flow.

The results of [3] for convection from a horizontal plane heat source, which revealed not only convective cells of hexagonal form, but also complex ornamental patterns of cells, indicate that the condition of filling of an infinite plane by a regular polygon is not a universal criterion for the existence of convective cellular structures of Benard's type. Thus, the question arises of whether group analysis results are applicable to similar structures and doubts are cast upon the universality of group methods as applied to convection problems considered here. This apparent contradiction is solved if we take into account that, strictly speaking, the generators describe the local properties of the medium, for example, the symmetry of flow inside a cell as a small-scale formation. Thus, the possibility of describing, in terms of these generators, regular plane pictures obtained by means of only two translational vectors is an exception rather than the rule. From this point of view there is no contradiction between group-theoretical and experimental [3] results.

A similar problem arises in describing convection from a horizontal heated cylinder [3], where in a small area around an individual buoying jet, the flow displays cylindrical symmetry but, at the same time, there is a translational invariance of the entire flow along the heat source.

Scale-Invariant Properties of Convective Flows. The dependences of the spatial and temporal scales of convective structures on the power input and on the initial temperature- and salt-stratification scales is of great interest.

We dwell first on the dependence of the scales on the power input. For this, we use a linear combination of generators of the form

$$G \cdot \partial = G_1 \cdot \partial + a\,(\Lambda_T)\, G_\chi \cdot \partial + b\,(\Lambda_S)\, G_{k_S} \cdot \partial. \tag{13}$$

Selecting the appropriate functions $a(\Lambda_T)$, $b(\Lambda_S)$ and $\varphi_\chi(\xi)$, $\varphi_{k_S}(\xi)$, one can obtain the operator $G \cdot \partial$ in the form

$$\begin{gathered} G \cdot \partial = x \cdot \partial_x + y \cdot \partial_y + z \cdot \partial_z + 2t \cdot \partial_t - (2p + P\,(z,t)) \cdot \partial_p \\ -\,(3m_\gamma + M_\gamma\,(z,t)) \cdot \partial_{m_\gamma} - (3m_\delta + M_\delta\,(z,t)) \cdot \partial_{m_\delta} - u \cdot \partial_u - v \cdot \partial_v - w \cdot \partial_w. \end{gathered} \tag{14}$$

The following step involves application of generator (14) to system (5) with a heat source. The explicit form of transformations of variables that are governed by generator (14) is obtained by integration of the Lie equations corresponding to this generator and having the form

$$\begin{gathered} \frac{d\tilde{x}}{d\xi} = \tilde{x}, \quad \frac{d\tilde{y}}{d\xi} = \tilde{y}, \quad \frac{d\tilde{z}}{d\xi} = \tilde{z}, \quad \frac{d\tilde{t}}{d\xi} = 2\tilde{t}, \quad \frac{d\tilde{u}}{d\xi} = -\tilde{u}, \quad \frac{d\tilde{v}}{d\xi} = -\tilde{v}, \quad \frac{d\tilde{w}}{d\xi} = -\tilde{w}, \\ \frac{d\tilde{p}}{d\xi} = -2\tilde{p} - P(\tilde{z},\tilde{t}), \qquad \frac{d\tilde{m}_\gamma}{d\xi} = -3\tilde{m}_\gamma - M_\gamma(\tilde{z},\tilde{t}), \qquad \frac{d\tilde{m}_\delta}{d\xi} = -3\tilde{m}_\delta - M_\delta(\tilde{z},\tilde{t}) \end{gathered}$$

(the tilde denotes the new variables and ξ is a group parameter). In addition, the that condition $\xi = 0$ is a stationary point of transformation should be satisfied, i.e.,

$$\tilde{x}(\xi)\Big|_{\xi=0} = x, \qquad \tilde{y}(\xi)\Big|_{\xi=0} = y,$$

etc., over all variables.

The next step in using group analysis results consists in replacing the old variable by new ones in system (5). The rules for the replacement are obtained after integration of the Lie equations, which gives the following transformation laws for the variables:

$$\tilde{x} = kx, \quad \tilde{y} = ky, \quad \tilde{z} = kz, \quad \tilde{t} = k^2 t, \quad \tilde{u} = k^{-1}u, \quad \tilde{v} = k^{-1}v, \quad \tilde{w} = k^{-1}w,$$

$$\tilde{p} = k^{-2}\Big[p - \int_1^{\ln k} P\left(\lambda z, \lambda^2 t\right)\lambda\, d\lambda\Big], \qquad \tilde{m}_{\gamma,\delta} = k^{-3}\Big[m_{\gamma,\delta} - \int_1^{\ln k} M_{\gamma,\delta}\left(\lambda z, \lambda^2 t\right)\lambda^2\, d\lambda\Big].$$

Here $k = e^{\xi}$; the derivatives with respect to spatial coordinates and time are transformed as

$$\frac{\partial}{\partial x} = k\frac{\partial}{\partial \tilde{x}}, \qquad \ldots, \qquad \frac{\partial}{\partial t} = k^2\frac{\partial}{\partial \tilde{t}},$$

and a term containing the source as

$$F(x,y,z,t) = k^{3-d}F(\tilde{x},\tilde{y},\tilde{z},\tilde{t})$$

(d is the dimensional parameter of the source).

Substitution of the resulting relationships into the equation of system (5) for transfer of the quantity m_γ yields

$$\frac{\partial \tilde{m}_\gamma}{\partial \tilde{t}} + \tilde{\mathbf{u}}\cdot\tilde{\nabla}\tilde{m}_\gamma + \tilde{w}\left(\frac{\tilde{T}^\gamma}{\tilde{\Lambda}_T} + \frac{\tilde{S}^\gamma}{\tilde{\Lambda}_S}\right) = k_\gamma\tilde{\Delta}\tilde{m}_\gamma + k_{\delta\gamma}\tilde{\Delta}\tilde{m}_\delta + k^{-2-d}\tilde{F}(\tilde{x},\tilde{y},\tilde{z},\tilde{t}),$$

where $\tilde{\nabla}$ and $\tilde{\Delta}$ are nabla and delta operators in the new variables.

The scales of dynamic structures vary with variation in the source intensity. Therefore, the invariance of (5) with change in source intensity can be achieved only by compensation of this change by the coefficient of the source term. Let the intensity of the source increase by q times. Then for the invariance of the equations it is necessary that the following relationship be fulfilled:

$$k = q^{1/(2+d)}.$$

In the general case of a heat source with dimensionality d, which can be noninteger (for example, if the source has the form of a fractal Koch's figure), the dependence of the spatial scale on the power input is representable as

$$L = L_0(\nu, \chi, k_S, \alpha, \beta, \ldots)F_0^{1/(2+d)}$$

(L_0 is a quantity independent of the source intensity F_0).

We use this relationship for special cases.

Point Heat Source. The source function is given by $F(\mathbf{R},t) = F_0\delta(x)\delta(y)\delta(z)\theta(t)$. Then

$$L = L_0(\nu, \chi, k_S, \alpha, \beta, \ldots)F_0^{1/2}.$$

Linear Heat Source. In the general case, $F(\mathbf{R},t) = F_0\delta(x)\delta(z - y\tan\gamma)\theta(t)$, where γ is the slope of the source to the horizon. As a result, the following relationship holds:

$$L = L_0(\nu, \chi, k_S, \alpha, \beta, \ldots)F_0^{1/3}.$$

Plane Heat Source. In the general case, the source can be given by $F(\mathbf{R},t) = F_0\delta(z + a\cdot x + b\cdot y)\theta(t)$, where a and b determine the spatial orientation of the plane. Then, with a change in power, the scale changes:

$$L = L_0(\nu, \chi, k_S, \alpha, \beta, \ldots)F_0^{1/4}.$$

In the case of a point source the results of [13] agrees with the result of this paper.

If an admixture source rather than a heat source is placed in a medium it turns out that the same dependences as for a heat source hold. Thus, if the buoyancy of the admixture is negative (salt, sugar, etc.), the flow pattern is specularly reflected from the plane $z = 0$, as was shown in [10].

In the study of the relationships between the dimensions of the structural elements and the initial temperature and concentrational stratifications, one of the main problems is that of finding those functions of physical fields whose spatial scales vary in proportional to Λ_T^a and Λ_S^b (a and b are arbitrary numbers).

For this, we shall compose a linear combination of the generators

$$G \cdot \partial = G_1 \cdot \partial + AG_\chi \cdot \partial + BG_{k_S} \cdot \partial, \tag{15}$$

where A and B are numbers that, in the general case, satisfy the relationships

$$4\Lambda_T + A\frac{\Lambda_T^2}{\chi T^0} J'_{\chi t} = \frac{\Lambda_T}{a}, \qquad 4\Lambda_S + B\frac{\Lambda_S^2}{k_S S^0} J'_{k_S t} = \frac{\Lambda_S}{b}. \tag{16}$$

The fulfillment of relationships (16) automatically ensures derivation of the sought-for relationships.

Integration of (15) makes it possible to introduce the function

$$\Psi_\lambda = m_\lambda - \frac{1}{3}\left[k^{\lambda/2} J'_{\chi_z} + k^{-\lambda/2} J'_{k_S z}\right], \tag{17}$$

whose spatial scales are proportional to Λ_T^a and Λ_S^b with the simultaneous scale-invariance of the function with a change in Λ_T and Λ_S.

As follows from (8), J_χ and J_{k_S} are, in the general case, functions of the form

$$\Phi(\lambda, z, t) = \int_{-\infty}^{\infty} \varphi(\xi, \Lambda_T, \Lambda_S) e^{\pm i\xi z - \xi^2 \lambda t}\, d\xi. \tag{18}$$

We consider the simplest case where the governing function $\varphi(\xi, \Lambda_T, \Lambda_S)$ is uniform for all J_λ and has the form

$$\varphi(\xi, \Lambda_T, \Lambda_S) = -\frac{1}{2}\delta''(\xi) \tag{19}$$

[$\delta(\xi)$ is a Dirac's delta function].

Substituting successively (19) into (18) and the results of integration into (17) and passing over to the initial physical fields of temperature T and salinity S, we obtain the function Ψ_λ in explicit form:

$$\Psi_\lambda = \left(\frac{\chi}{k_S}\right)^{\lambda/2} \left(\alpha\bar{T} - \frac{\chi}{\chi - k_S}\right) - \left(\frac{k_S}{\chi}\right)^{\lambda/2} \left(\beta\bar{S} - \frac{k_S}{\chi - k_S}\right) \tag{20}$$

(the designations $\bar{T}$ and $\bar{S}$ were introduced earlier and describe the total temperature and salinity fields).

For $\lambda = 0$ and $\lambda = 1$, from (20), we obtain

$$\Psi_0 = 1 + \beta\bar{S} - \alpha\bar{T} = H, \qquad \Psi_1 = -\chi - k_S + \chi\alpha\bar{T} - k_S\beta\bar{S} = G,$$

where H and G are the reduced density and the kinetic dilation potential, which were used in [13] as field variables.

Depending on power input, three types of layered flow from a point source were distinguished in [16]. For the first type of the flow, the limiting height of the structure is achieved immediately owing to a primary convective jet which arises directly from the source and, for a second type, secondary convective cells which sequentially form above the primary dome are taken into account. A third type is intermediate between the above two. In all cases, as was shown in experiments, the structure height is proportional to $F_0^{1/2}$. We consider the formation of flow taking into account that the height of convective cells (thickness of layers) is proportional to $F_0^{1/7}$ [16].

As the heat source is switched on, the overheating of the fluid located in the nearest vicinity reaches tens of degrees. As follows from [13], the primary heat jet height is proportional to $F_0^{1/2}$. Let the structure height be defined as $L_{st} = L_0 F_0^{1/2}$, the the primary jet height as $L_p = L_1 F_0^{1/2}$, and the convective cell height as $L_c = L_2 F_0^{1/7}$, where L_0, L_1, and L_2 are constant quantities which are determined by the fluid parameters.

Let $n(F_0)$ be the dependence of the number of convectie cells (layers) on power. Then, in the first case (at the center of the structure), we have $L_{st_1} = L_{p_1}$. On the others hand, $L_{st_1} = n(F_0)L_c$. As a result,

$$n(F_0) = \frac{L_0}{L_2} F_0^{5/14}.$$

In the second case, $L_{st_2} = L_{p_2} + n(F_0)L_c$, and hence

$$n(F_0) = \frac{L_0 - L_1}{L_2} F_0^{5/14}.$$

In the intermediate case, where some layers are formed near the primary jet and the other are generated by secondary sources above the primary jet, a similar analysis shows that the total number of layers is also proportional to $F_0^{5/14}$.

Since the number of layers n is a natural quantity, the dependence $n(F_0)$ has a stepwise character with an envelope behaving as $F_0^{5/14}$. Hence, it also follows that there are some critical power values for which the number of layers is increased by one. These critical values are defined by the relationship $F_{0n}^* = (n/N_0)^{14/5}$ (N_0 is a constant determined by the medium's parameters, $n = 1, 2, \ldots$).

The resulting dependence $n(F_0) \sim F_0^{5/14}$ is universal for all power values where the layered convection regime is realized.

In this investigation it was assumed that the kinetic coefficients of the medium are constant and independent of temperature and the admixture concentration. This is not valid for all fluid and gaseous media. Therefore, in a more thorough study of dynamic structures it is necessary to take into account the properties of particular media more precisely and reflect this fact even in the initial hydrodynamic equations.

Analysis of the system of equations of thermoconcentrational convection for a stratified fluid shows that the introduction of generalized density disturbances facilitates accomplishment of group analysis and can be recommended for numerical modeling of convection processes.

The introduced generalized density disturbances are shown to be universal in the sense that they describe all possible combinations of temperature and concentration of admixtures that are obtained in the system of thermoconcentrational convection equation by linear combination.

The resulting partially symmetrized system involves field variables (temperature and concentration of admixtures) and the corresponding kinetic coefficients (thermal diffusivity and salt diffusivity).

An infinite-dimensional group of transformations that reflects the symmetry properties of flows over various types of sources is determined.

The splitting of variation scales for various variables (velocity, temperatures, and salt) is a universal property of thermoconcentrational convection.

Without employing additional hypotheses, dependences of the global scale of the structure and its individual elements on power input (buoyancy flow), which are in agreement with experimental data for convection, are obtained.

It shown that the infinite set of functions that display invariant properties with change in the initial stratification scale includes the reduced fluid density and the kinetic dilation potential.

This work was supported by the Russian Foundation for Fundamental Research (Grant 93-05-8291).

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